

# Characterizations of Fredholm Pairs and Chains in Hilbert Spaces

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## Abstract

In this work characterizations of Fredholm pairs and chains of Hilbert space operators are given. Following a well-known idea of several variable operator theory in Hilbert spaces, the aforementioned objects are characterized in terms of Fredholm linear and bounded maps. Furthermore, as an application of the main results of this work, direct proofs of the stability properties of Fredholm pairs and chains in Hilbert spaces are obtained.

*AMS 2000 Subject Classification:* Primary 47A13; Secondary 47A53, 47A55.

*Key words and phrases:* Fredholm pairs and chains, Fredholm self-adjoint operators, index, stability properties.

## 1. INTRODUCTION

In multiparameter operator theory there are objects that in the frame of Hilbert spaces can be described in terms of suitable linear and continuous maps. It is important to remark that this reduction of the complexity of certain problems from several variable operator theory to the classical setting of one single operator allows not only to recover properties and techniques from one dimensional operator theory, but also to simplify proofs. For instance, as regard Hilbert space complexes, the exactness (resp. Fredholmness) of such an object can be determined by the invertibility (resp. the Fredholmness) of a Hilbert space operator, see [11], [5] and [12, Chap. III]. Furthermore, this connection between one and several variable operator theory was for the first time developed to give a characterization of the Taylor joint spectrum in Hilbert space in terms of the invertibility of a single Hilbert space linear and bounded map, see [8] and [9]. In addition, the application of this idea led to many new results in the area under consideration, see for example [8], [9], [3], [4], [10], [11], [5] and [12].

On the other hand, Fredholm pairs were studied in the works [1] and [2], where the main stability properties of such objects were also proved. Roughly speaking, the aforementioned pairs consist in an extension of the notion of Fredholm operator to multiparameter spectral theory, which is closely related to the concept of Fredholm Banach space complex, see [1] and [2]. However, Fredholm pairs have not been studied in the frame of Hilbert spaces yet. In this work two characterizations of these objects are given. In fact, following the idea mentioned in the first paragraph, Fredholm pairs are characterized in terms of Fredholm Hilbert space

linear and bounded maps. Moreover, these characterizations will be applied to directly prove the stability properties of the objects under consideration, as well as to study dual Fredholm pairs, see section 4.

A generalization of the concept of Fredholm Banach spaces complex is the notion of Fredholm chain, which is closely related to the one of Fredholm pair, see [6]. In this work, thanks to the above-mentioned results on Fredholm pairs, two characterizations of Fredholm chains in Hilbert spaces will be given. In fact, these objects will be characterized in terms of Fredholm Hilbert space operators. Furthermore, the stability properties of the objects under consideration will be proved and dual Fredholm chains will be studied, see section 6.

The article is organized as follows. In the next section some definitions and facts needed for the present work are reviewed. In sections 3 and 4 the main results of this work are proved. In addition, in section 4 dual Fredholm pairs are also studied. In section 5, as an application of the characterization studied in section 4, the stability properties of Fredholm pairs in Hilbert spaces are proved. Finally, in section 6, Fredholm chains of Hilbert space operators are considered. In fact, these objects are characterized, their stability properties are proved, and dual Fredholm chains are studied.

## 2. PRELIMINARY DEFINITIONS AND FACTS

Since all the operators considered in this article will be defined on Hilbert spaces, all the definitions and facts reviewed will be restricted to this class of spaces and maps. For a general presentation, see the works [1], [2] and [6].

From now on,  $H_1$  and  $H_2$  denote two Hilbert spaces,  $L(H_1, H_2)$  the algebra of all linear and continuous maps defined on  $H_1$  with values in  $H_2$ , and  $K(H_1, H_2)$  the closed ideal of all compact operators of  $L(H_1, H_2)$ . As usual, when  $H_1 = H = H_2$ ,  $L(H, H)$  and  $K(H, H)$  are denoted by  $L(H)$  and  $K(H)$  respectively. For every  $S \in L(H_1, H_2)$ , the null space of  $S$  is denoted by  $N(S) = \{x \in H_1 : S(x) = 0\}$ , and the range of  $S$  by  $R(S) = \{y \in H_2 : \exists x \in H_1 \text{ such that } y = S(x)\}$ . Next follows the definition of Fredholm pair, see for instance [1].

*Definition 2.1.* Let  $H_1$  and  $H_2$  be two Hilbert spaces. Let  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  be such that the following dimensions are finite:

$$\begin{aligned} a &:= \dim N(S)/(N(S) \cap R(T)), \quad b := \dim R(T)/(N(S) \cap R(T)), \\ c &:= \dim N(T)/(N(T) \cap R(S)), \quad d := \dim R(S)/(N(T) \cap R(S)). \end{aligned}$$

A pair  $(S, T)$  with the above properties is called a Fredholm pair (see [1]).

Let  $P(H_1, H_2)$  denote the set of all Fredholm pairs. If  $(S, T) \in P(H_1, H_2)$ , then the index of  $(S, T)$  is defined by the equality

$$\text{ind}(S, T) := a - b - c + d.$$

Before going on, several properties of Fredholm pairs are recalled, see [1].

*Remark 2.2.* First of all, observe that if  $S \in L(H_1, H_2)$  is a Fredholm operator, then  $(S, 0)$  is a Fredholm pair. Furthermore,  $\text{ind } S = \text{ind } (S, 0)$ . Consequently, the definition of Fredholm pair extends the notion of Fredholm operator to several variable operator theory.

In second place, note that if  $(S, T) \in P(H_1, H_2)$ , then  $(T, S) \in P(H_2, H_1)$  and

$$\text{ind } (T, S) = -\text{ind } (S, T).$$

Finally, if  $(S, T) \in P(H_1, H_2)$ , then  $R(S)$  and  $N(T) + R(S)$  are closed subspaces in  $H_2$ . Similarly,  $R(T)$  and  $N(S) + R(T)$  are closed subspaces in  $H_1$ .

Next follows the definition of Fredholm chains, see for instance [6].

*Definition 2.3.* A Fredholm chain  $(H, \delta)$  is a sequence of spaces and maps

$$0 \rightarrow H_n \xrightarrow{\delta_n} H_{n-1} \rightarrow \dots \rightarrow H_1 \xrightarrow{\delta_1} H_0 \rightarrow 0,$$

where  $H_p$  are Hilbert spaces, and  $\delta_p \in L(H_p, H_{p-1})$  are bounded operators such that

$$N(\delta_p)/(N(\delta_p) \cap R(\delta_{p+1})) \text{ and } R(\delta_{p+1})/(N(\delta_p) \cap R(\delta_{p+1}))$$

are finite dimensional subspaces of  $H_p$ ,  $p = 0, \dots, n$ . Formally, it is assumed that  $H_p = 0$  and  $\delta_p = 0$ , for  $p < 0$  and  $p \geq n+1$ .

Given a Fredholm chain, it is possible to associate to it an index. In fact, if  $(H, \delta)$  is such an object, then define

$$\begin{aligned} \text{ind } (H, \delta) = & \sum_{p=0}^n (-1)^p (\dim N(\delta_p)/(N(\delta_p) \cap R(\delta_{p+1})) \\ & - \dim R(\delta_{p+1})/(N(\delta_p) \cap R(\delta_{p+1}))), \end{aligned}$$

see [6].

Recall that in [6] it was introduced the more general concept of semi-Fredholm chain. However, since the main concern of this article consists in Fredholm objects, only Fredholm chains will be considered. Furthermore, observe that since  $\dim R(\delta_{p-1}\delta_p)$  are finite dimensional,  $p = 1, \dots, n$ , a Fredholm chain  $(H, \delta)$  is a particular case of what in [7] was called an essential complex of Banach spaces.

In the following remark, the relationship between Fredholm pairs and chains is considered.

*Remark 2.4.* Consider  $(H, \delta)$  a sequence of spaces and maps

$$0 \rightarrow H_n \xrightarrow{\delta_n} H_{n-1} \rightarrow \dots \rightarrow H_1 \xrightarrow{\delta_1} H_0 \rightarrow 0,$$

where  $H_p$  are Hilbert spaces, and  $\delta_p \in L(H_p, H_{p-1})$  are bounded operators. In addition, assume that  $H_p = 0$  and  $\delta_p = 0$ , for  $p < 0$  and  $p \geq n+1$ .

Next, associate to this sequences the Hilbert spaces

$$H_1 = \bigoplus_{p=2k} H_p, \quad H_2 = \bigoplus_{p=2k+1} H_p,$$

and the maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  defined as

$$S = \bigoplus_{p=2k} \delta_p, \quad T = \bigoplus_{p=2k+1} \delta_p,$$

where  $H_p = 0$  and  $\delta_k = 0$ , when  $p < 0$  and  $p \geq n+1$ .

Since

$$R(ST) = \bigoplus_{p=2k} R(\delta_p \delta_{p+1}) \in L(H_2), \quad R(TS) = \bigoplus_{p=2k+1} R(\delta_p \delta_{p+1}) \in L(H_1),$$

$\dim R(ST)$  and  $\dim R(TS)$  are finite dimensional if and only if  $\dim R(\delta_p \delta_{p+1}) = \dim R(\delta_{p+1}) / (N(\delta_p) \cap R(\delta_{p+1}))$  are finite dimensional,  $p = 0, \dots, n$ .

Furthermore, since it is clear that

$$N(S) / (N(S) \cap R(T)) = \bigoplus_{p=2k} N(\delta_p) / (N(\delta_p) \cap R(\delta_{p+1})),$$

$$N(T) / (N(T) \cap R(S)) = \bigoplus_{p=2k+1} N(\delta_p) / (N(\delta_p) \cap R(\delta_{p+1})),$$

the sequence  $(H, \delta)$  is a Fredholm chain if and only if  $(S, T)$  is a Fredholm pair.

Finally, in this case a straightforward calculation shows that

$$\text{ind } (H, \delta) = \text{ind } (S, T).$$

### 3. THE FIRST CHARACTERIZATION

In this section, the first characterization of Fredholm pairs of Hilbert space operators is given. To this end, the argument develops ideas of [12, Chap. III, Crollary 7.4], see also [8, Theorem 2.1] and [5, Proposition 2.1].

In order to prove the main result of this section, several previous propositions are needed. In first place, some preparation is presented.

Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$ , two operators such that  $\dim R(ST)$  and  $\dim R(TS)$  are finite dimensional. Define the Hilbert space  $H$  as the orthogonal direct sum of  $H_i$ ,  $i=1, 2$ , that is  $H = H_1 \oplus H_2$ . Now well, if  $U \in L(H)$  is the linear and continuous map

$$U = \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix},$$

then consider the self-adjoint operator  $V = U + U^* \in L(H)$ . The following proposition is the first step to the first characterization of Fredholm pairs in Hilbert spaces.

**PROPOSITION 3.1.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider two bounded and linear maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  such that  $R(TS)$  and  $R(ST)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Then, if  $U$  and  $V$  are the operators defined above, the following statements are equivalent:*

- i)  $V = U + U^*$  is a Fredholm operator,
- ii)  $V^2$  is a Fredholm operator,
- iii)  $UU^* + U^*U$  is a Fredholm operator,
- iv)  $TT^* + S^*S \in L(H_1)$  and  $SS^* + T^*T \in L(H_2)$  are Fredholm operators.

*Proof.* First of all, it is clear that  $V$  is a Fredholm operator if and only if  $V^2$  is. In addition, note that  $V^2 = U^2 + U^{*2} + UU^* + U^*U$ , where

$$U^2 = \begin{pmatrix} TS & 0 \\ 0 & ST \end{pmatrix}, \quad U^{*2} = \begin{pmatrix} (ST)^* & 0 \\ 0 & (TS)^* \end{pmatrix},$$

and

$$UU^* + U^*U = \begin{pmatrix} TT^* + S^*S & 0 \\ 0 & SS^* + T^*T \end{pmatrix}.$$

Now well, since  $\dim R(ST)$  and  $\dim R(TS)$  are finite dimensional,  $\dim R(U^2)$  and  $\dim R(U^{*2})$  also are finite dimensional. Therefore,  $V^2$  is a Fredholm operator if and only if  $UU^* + U^*U$  is a Fredholm operator, which is equivalent to the fact that  $TT^* + S^*S$  and  $SS^* + T^*T$  are Fredholm operators.  $\square$

The following remarks are needed to prove the main result of the present section.

*Remark 3.2.* Let  $H$  be a Hilbert space and consider  $T \in L(H)$  such that  $T^2 = 0$ , that is  $R(T) \subseteq N(T)$ . Then, according to [12, Chap. III, Lemma 7.3],  $R(T) = N(T)$  if and only if the self-adjoint operator  $T + T^*$  is invertible on  $H$ . Now well, suppose that  $R(T)$  is closed. Then, according to the argument in the aforementioned Lemma, it is easy to prove that  $\dim N(T)/R(T)$  is finite dimensional if and only if  $T + T^*$  is a Fredholm self-adjoint operator.

*Remark 3.3.* Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider two bounded and linear maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  such that  $F_1 = R(TS)$  and  $F_2 = R(ST)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Decompose  $H_1$  and  $H_2$  as the following orthogonal direct sum:

$$H_1 = F_1 \oplus \mathcal{H}_1, \quad H_2 = F_2 \oplus \mathcal{H}_2,$$

where  $\mathcal{H}_i = F_i^\perp$ ,  $i=1, 2$ .

Next consider the operators  $\mathcal{S} \in L(\mathcal{H}_1, \mathcal{H}_2)$  and  $\mathcal{T} \in L(\mathcal{H}_2, \mathcal{H}_1)$  defined by

$$\mathcal{S} = P_2 \circ (S \mid \mathcal{H}_1), \quad \mathcal{T} = P_1 \circ (T \mid \mathcal{H}_2),$$

where  $P_i: H_i \rightarrow \mathcal{H}_i$  is the orthogonal projection onto  $\mathcal{H}_i$ ,  $i=1, 2$ . It is clear that

$$\mathcal{S} \circ \mathcal{T} = 0, \quad \mathcal{T} \circ \mathcal{S} = 0.$$

Now well, according to [1, Remark 2.1], it is easy to prove that  $(S, T)$  is a Fredholm pair if and only if  $(\mathcal{S}, \mathcal{T})$  is a Fredholm pair, which is equivalent to the fact that  $\dim N(\mathcal{S})/R(\mathcal{T})$  and  $\dim N(\mathcal{T})/R(\mathcal{S})$  are finite dimensional.

Furthermore, in this case

$$\text{ind}(S, T) = \text{ind}(\mathcal{S}, \mathcal{T}) - \dim R(ST) + \dim R(TS).$$

The next result is the key step for the first characterization of Fredholm pairs in Hilbert spaces.

**PROPOSITION 3.4.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider two bounded and linear maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  such that  $R(TS)$  and  $R(ST)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Then,  $(S, T)$  belongs to  $P(H_1, H_2)$  if and only if  $TT^* + S^*S \in L(H_1)$  and  $SS^* + T^*T \in L(H_2)$  are Fredholm operators.*

*Proof.* First of all, consider the finite dimensional subspaces  $F_1 = R(TS)$  and  $F_2 = R(ST)$ , and decompose  $H_1$  and  $H_2$  as in Remark 3.3, that is

$$H_1 = F_1 \oplus \mathcal{H}_1, \quad H_2 = F_2 \oplus \mathcal{H}_2,$$

where  $\mathcal{H}_i = F_i^\perp$ ,  $i=1, 2$ .

In addition, consider the operators  $\mathcal{S} \in L(\mathcal{H}_1, \mathcal{H}_2)$  and  $\mathcal{T} \in L(\mathcal{H}_2, \mathcal{H}_1)$  defined in Remark 3.3.

Now suppose that  $(S, T)$  is a Fredholm pair. Then, according to Remark 3.3 or to [1, Remark 2.1],  $(\mathcal{S}, \mathcal{T})$  is a Fredholm pair, that is  $\dim N(\mathcal{S})/R(\mathcal{T})$  and  $\dim N(\mathcal{T})/R(\mathcal{S})$  are finite dimensional.

Now well, if  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , and if

$$\mathcal{U} = \begin{pmatrix} 0 & \mathcal{T} \\ \mathcal{S} & 0 \end{pmatrix},$$

then it is easy to prove that  $R(\mathcal{U})$  is a closed subspace of  $\mathcal{H}$ ,  $\mathcal{U}^2 = 0$ , and  $\dim N(\mathcal{U})/R(\mathcal{U})$  is finite dimensional. Consequently, according to Remark 3.2,  $\mathcal{V} = \mathcal{U} + \mathcal{U}^*$  is also a Fredholm operator, and according to Proposition 3.1,  $\mathcal{T}\mathcal{T}^* + \mathcal{S}^*\mathcal{S} \in L(\mathcal{H}_1)$  and  $\mathcal{S}\mathcal{S}^* + \mathcal{T}^*\mathcal{T} \in L(\mathcal{H}_2)$  are Fredholm operators.

On the other hand, if  $\mathcal{S}$  (resp.  $\mathcal{T}$ ) is extended to  $H_1$  (resp.  $H_2$ ) by setting  $\mathcal{S} \mid F_1 \equiv 0$  (resp.  $\mathcal{T} \mid F_2 \equiv 0$ ), then it is clear that there are operators  $S_1 \in L(H_1, H_2)$  and  $T_1 \in L(H_2, H_1)$  such that  $R(S_1)$  and  $R(T_1)$  are finite dimensional and

$$S = \mathcal{S} + S_1, \quad T = \mathcal{T} + T_1,$$

where  $\mathcal{S} \in L(H_1, H_2)$  and  $\mathcal{T} \in L(H_2, H_1)$  also denote the extension of  $\mathcal{S} \in L(\mathcal{H}_1, \mathcal{H}_2)$  and  $\mathcal{T} \in L(\mathcal{H}_2, \mathcal{H}_1)$  respectively.

Now well, a straightforward calculation proves that there are two operators  $K_1 \in L(H_1)$  and  $K_2 \in L(H_2)$  whose ranges are finite dimensional and such that

$$TT^* + S^*S = \mathcal{T}\mathcal{T}^* + \mathcal{S}^*\mathcal{S} + K_1, \quad SS^* + T^*T = \mathcal{S}\mathcal{S}^* + \mathcal{T}^*\mathcal{T} + K_2.$$

Therefore,  $TT^* + S^*S$  and  $SS^* + T^*T$  are Fredholm operators.

Conversely, according to the previous argument, if  $TT^* + S^*S$  and  $SS^* + T^*T$  are Fredholm operators, then it is clear that  $\mathcal{T}\mathcal{T}^* + \mathcal{S}^*\mathcal{S} \in L(\mathcal{H}_1)$  and  $\mathcal{S}\mathcal{S}^* + \mathcal{T}^*\mathcal{T} \in L(\mathcal{H}_2)$  are Fredholm operators. Furthermore, according to Proposition 3.1,  $\mathcal{V} = \mathcal{U} + \mathcal{U}^* \in L(\mathcal{H})$  is a Fredholm operator, and according to Remark 3.2,  $\dim N(\mathcal{U})/R(\mathcal{U})$  is finite dimensional.

Now well,

$$N(\mathcal{U}) = N(\mathcal{S}) \oplus N(\mathcal{T}), \quad R(\mathcal{U}) = R(\mathcal{T}) \oplus R(\mathcal{S}).$$

Consequently,  $(\mathcal{S}, \mathcal{T})$  is a Fredholm pair. However, since  $R(ST)$  and  $R(TS)$  are finite dimensional subspaces of  $H_2$  and  $H_1$  respectively, according to Remark 3.3 or to [1, Remark 2.1],  $(S, T)$  is a Fredholm pair.  $\square$

In the next theorem the first characterization of Fredholm pairs in Hilbert spaces is presented.

**THEOREM 3.5.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider two bounded and linear maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  such that  $R(TS)$  and  $R(ST)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Then, with the notations of Proposition 3.1, the following statements are equivalent:*

- i)  $(S, T)$  is a Fredholm pair
- ii)  $V = U + U^*$  is a Fredholm operator,
- iii)  $V^2$  is a Fredholm operator,
- iv)  $UU^* + U^*U$  is a Fredholm operator,
- v)  $TT^* + S^*S \in L(H_1)$  and  $SS^* + T^*T \in L(H_2)$  are Fredholm operators.

*Proof.* It is a consequence of Propositions 3.1 and 3.4.  $\square$

#### 4. THE SECOND CHARACTERIZATION

In Theorem 3.5 the condition of being a Fredholm pair is expressed in terms of Fredholm self-adjoint operators. This formulation has the disadvantage that the index of a Fredholm pair can not be related to the index of any of the linear and continuous maps considered in the aforementioned theorem. In this section, however, it is proved another characterization of Fredholm pairs in Hilbert spaces which precisely has the advantage that the index of a Fredholm pair is expressed in terms of the index of a Fredholm operator defined between two Hilbert spaces. In addition, as a first application of this characterization, dual Fredholm pairs will be studied. On the other hand, the argument of the main result of this section develops ideas of [11, Theorem 1.2] and [12, Chap. III, Theorem 7.1].

First of all, in order to prove the second characterization of Fredholm pairs in Hilbert spaces, some preparation is needed.

**PROPOSITION 4.1.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider two operators  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  such that  $TS = 0$  and  $ST = 0$ . Then, the following statements are equivalent:*

- i)  $\dim N(S)/R(T)$  and  $\dim N(T)/R(S)$  are finite dimensional,
- ii)  $S + T^* \in L(H_1, H_2)$  is a Fredholm operator,
- iii)  $T + S^* \in L(H_2, H_1)$  is a Fredholm operator.

Furthermore, in this case  $(S, T)$  is a Fredholm pair and

$$\text{ind } (S, T) = \text{ind } (S + T^*) = - \text{ind } (T + S^*).$$

*Proof.* First of all, it is clear that  $S + T^*$  is a Fredholm operator if and only if  $T + S^*$  is, and in this case  $\text{ind } (T + S^*) = - \text{ind } (S + T^*)$ .

Next, note that if  $\dim N(S)/R(T)$  and  $\dim N(T)/R(S)$  are finite dimensional, then  $R(S)$  and  $R(T)$  are closed subspaces of  $H_2$  and  $H_1$  respectively. Conversely, if  $S + T^*$  is a Fredholm operator, then  $R(S)$  and  $R(T)$  are closed.

In fact, consider  $(x_n)_{n \in \mathbb{N}} \subseteq N(S)^\perp$  such that  $(S(x_n))_{n \in \mathbb{N}}$  converges to  $v \in H_2$ . Since the closure of  $R(T)$  is contained in  $N(S)$ ,

$$N(S)^\perp \subseteq \overline{R(T)}^\perp = R(T)^\perp \subseteq N(T^*).$$

In particular,  $(S + T^*)(x_n) = S(x_n)$ , and  $v \in R(S + T^*)$ . Thus, there is  $x \in H_1$  such that  $(S + T^*)(x) = v$ . However,  $(S(x_n - x))_{n \in \mathbb{N}}$  converges to  $T^*(x)$ . Then,

$$T^*(x) \in R(T^*) \cap \overline{R(S)} \subseteq R(T^*) \cap N(T) \subseteq R(T^*) \cap R(T^*)^\perp = 0.$$

Consequently,  $v = S(x) \in R(S)$ .

Since  $T + S^*$  is Fredholm if and only if  $T + S^*$  is, a similar argument proves that  $R(T)$  is a closed subspace of  $H_1$ .

Now well, since it has been proved that in the conditions of the proposition  $R(S)$  and  $R(T)$  are always closed subspaces of  $H_2$  and  $H_1$  respectively, in the rest of the proof it will be assumed this property for  $S$  and  $T$ .

Decompose, then,  $H_1$  and  $H_2$  as the orthogonal direct sum of the following subspaces:

$$H_1 = (R(T) \oplus N_1) \oplus L_1, \quad H_2 = (R(S) \oplus N_2) \oplus L_2,$$

where

$$R(T) \oplus N_1 = N(S), \quad L_1 = N(S)^\perp, \quad R(S) \oplus N_2 = N(T), \quad L_2 = N(T)^\perp.$$

Now well, according to the above orthogonal direct sum, and since  $N(T^*) = R(T)^\perp = N_1 \oplus L_1$  and  $R(T^*) = N(T)^\perp = L_2$ , it is clear that the operator  $S + T^*$  can be presented in the following matricial form:

$$\begin{pmatrix} 0 & 0 & \mathcal{S} \\ 0 & 0 & 0 \\ \mathcal{T}^* & 0 & 0 \end{pmatrix},$$

where

$$\mathcal{S} = S|_{L_1} : L_1 \xrightarrow{\cong} R(S), \quad \mathcal{T} = T|_{L_2} : L_2 \xrightarrow{\cong} R(T).$$



Moreover, this matricial decomposition gives

$$N(S + T^*) = N_1, \quad R(S + T^*)^\perp = N_2.$$

Therefore,  $N(S)/R(T)$  and  $N(T)/R(S)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively if and only if  $S + T^*$  is a Fredholm operator.

In addition, in this case, since  $ST = 0$  and  $TS = 0$ , and since  $\dim N_1 = \dim N(S)/R(T)$  and  $\dim N_2 = \dim N(T)/R(S)$ ,  $(S, T) \in P(H_1, H_2)$  and

$$\text{ind } (S, T) = \text{ind } (S + T^*). \quad \square$$

Next follows the main result of the present section.

**THEOREM 4.2.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider two bounded and linear maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  such that  $R(TS)$  and  $R(ST)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Then, the following statements are equivalent:*

- i)  $(S, T)$  is a Fredholm pair,
- ii)  $S + T^* \in L(H_1, H_2)$  is a Fredholm operator,
- iii)  $T + S^* \in L(H_2, H_1)$  is a Fredholm operator.

Furthermore, in this case

$$\text{ind } (S, T) = \text{ind } (S + T^*) = - \text{ind } (T + S^*).$$

*Proof.* First of all, consider the finite dimensional subspaces  $F_1 = R(TS)$  and  $F_2 = R(ST)$ , and decompose  $H_1$  and  $H_2$  as in Remark 3.3, that is

$$H_1 = F_1 \oplus \mathcal{H}_1, \quad H_2 = F_2 \oplus \mathcal{H}_2,$$

where  $\mathcal{H}_i = F_i^\perp$ ,  $i=1, 2$ .

In addition, consider the operators  $\mathcal{S} \in L(\mathcal{H}_1, \mathcal{H}_2)$  and  $\mathcal{T} \in L(\mathcal{H}_2, \mathcal{H}_1)$  defined in Remark 3.3. Therefore, according to Remark 3.3 or to [1, Remark 2.1],  $(S, T)$  is a Fredholm pair if and only if  $(\mathcal{S}, \mathcal{T}) \in P(\mathcal{H}_1, \mathcal{H}_2)$ , equivalently,  $N(\mathcal{S})/R(\mathcal{T})$  and  $N(\mathcal{T})/R(\mathcal{S})$  are finite dimensional subspaces of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively. However, according to Proposition 4.1, this is equivalent to the fact that  $\mathcal{S} + \mathcal{T}^* \in L(\mathcal{H}_1, \mathcal{H}_2)$  is a Fredholm operator.

Next, as in Proposition 3.4, extend  $\mathcal{S}$  (resp.  $\mathcal{T}$ ) to  $H_1$  (resp.  $H_2$ ) by setting  $\mathcal{S} \mid F_1 \equiv 0$  (resp.  $\mathcal{T} \mid F_2 \equiv 0$ ), and denote this extension by  $S$  (resp.  $T$ ). In addition, again as in Proposition 3.4, consider operators  $S_1 \in L(H_1, H_2)$  and  $T_1 \in L(H_2, H_1)$  such that  $R(S_1)$  and  $R(T_1)$  are finite dimensional and

$$S = \mathcal{S} + S_1, \quad T = \mathcal{T} + T_1.$$

Now well, since  $F_1$  and  $F_2$  are finite dimensional subspaces of  $H_1$  and  $H_2$ ,  $\mathcal{S} + \mathcal{T}^* \in L(\mathcal{H}_1, \mathcal{H}_2)$  is a Fredholm operator if and only if  $S + T^* \in L(H_1, H_2)$  is a Fredholm bounded

a linear map. However, since  $R(S_1)$  and  $R(T_1^*)$  are finite dimensional subspaces of  $H_2$ , and since

$$S + T^* = (\mathcal{S} + \mathcal{T}^*) + (S_1 + T_1^*),$$

$\mathcal{S} + \mathcal{T}^* \in L(H_1, H_2)$  is a Fredholm operator if and only if  $S + T^*$  is a Fredholm linear and continuous map. Moreover, in this case

$$\text{ind } (S + T^*) = \text{ind } (\mathcal{S} + \mathcal{T}^*).$$

However, according to Proposition 4.1, for  $\mathcal{S} + \mathcal{T}^* \in L(H_1, H_2)$ ,

$$\text{ind } (\mathcal{S} + \mathcal{T}^*) = \text{ind } (\mathcal{S}, \mathcal{T}) - \dim R(ST) + \dim R(TS),$$

and according to Remark 3.3 or to [1, Remark 2.1],

$$\text{ind } (S + T^*) = \text{ind } (S, T).$$

Finally, it is clear that  $S + T^*$  is a Fredholm operator if and only if  $T + S^*$  is, and in this case,  $\text{ind } (S + T^*) = -\text{ind } (T + S^*)$ .  $\square$

As a first application of Theorem 4.2, the relationship between Fredholm pairs and adjoint operators in Hilbert spaces is studied. To this end, some preparation is needed.

Let  $H_i$ ,  $i=1, 2$ , be two Hilbert spaces,  $S$  belong to  $L(H_1, H_2)$  and  $T$  to  $L(H_2, H_1)$ . Denote  $(S, T)^* \equiv (T^*, S^*)$ , where  $S^*$  (resp.  $T^*$ ) is the adjoint map of  $S$  (resp.  $T$ ). Note that  $(S, T)^{**} = (T^*, S^*)^* = (S, T)$ .

**THEOREM 4.3.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider  $(S, T) \in L(H_1, H_2) \times L(H_2, H_1)$ . Then,  $(S, T)$  is a Fredholm pair if and only if  $(T^*, S^*)$  is. Furthermore, in this case*

$$\text{ind } (T^*, S^*) = -\text{ind } (S, T).$$

*Proof.* Since  $(T^*, S^*)^* = (S, T)$ , it is enough to prove the first part of the proposition.

First of all, note that since  $S^*T^* = (TS)^*$  and  $T^*S^* = (ST)^*$ , then, if  $(S, T) \in P(H_1, H_2)$ ,  $R(S^*T^*)$  and  $R(T^*S^*)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively.

Now well, according to Theorem 4.2, if  $(S, T) \in P(H_1, H_2)$ , then  $S^* + T \in L(H_2, H_1)$  is a Fredholm operator. Consequently, according to Theorem 4.2 again,  $(S^*, T^*) \in P(H_2, H_1)$  and

$$\text{ind } (S, T) = -\text{ind } (S^*, T^*).$$

However, according to Remark 2.2 or to the observation that follows [1, Definition 1.1],  $(T^*, S^*)$  is a Fredholm pair and

$$\text{ind } (S, T) = \text{ind } (T^*, S^*). \quad \square$$

## 5. STABILITY PROPERTIES

In this section, thanks to Theorem 4.2, the stability properties of Fredholm pairs in Hilbert spaces are proved in a direct way. In addition, note that the hypothesis in [1, Theorem 3.1], [1, Theorem 3.2] and [2, Theorem 4] can be weakened.

**THEOREM 5.1.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider  $(S, T) \in P(H_1, H_2)$ . Let  $S_1 \in L(H_1, H_2)$  and  $T_1 \in L(H_2, H_1)$  be two operators such that  $R(T_1 S_1)$  and  $R(S_1 T_1)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Then, there is an  $\epsilon > 0$  such that if  $\|S - S_1\| < \epsilon$  and  $\|T - T_1\| < \epsilon$ ,  $(S_1, T_1)$  is a Fredholm pair. Furthermore,*

$$\text{ind}(S, T) = \text{ind}(S_1, T_1).$$

*Proof.* First of all, according to Theorem 4.2,  $S + T^* \in L(H_1, H_2)$  is a Fredholm operator and  $\text{ind}(S, T) = \text{ind}(S + T^*)$ .

Now well, since

$$\|(S + T^*) - (S_1 + T_1^*)\| < \|S - S_1\| + \|T - T_1\|,$$

there is an  $\epsilon > 0$  such that if  $\|S - S_1\| < \epsilon$  and  $\|T - T_1\| < \epsilon$ , then  $S_1 + T_1^*$  is a Fredholm operator and  $\text{ind}(S + T^*) = \text{ind}(S_1 + T_1^*)$ .

However, according to Theorem 4.2,  $(S_1, T_1)$  is a Fredholm pair and

$$\text{ind}(S, T) = \text{ind}(S_1, T_1). \quad \square$$

**THEOREM 5.2.** *Let  $H_1$  and  $H_2$  be two Hilbert spaces and consider  $(S, T) \in P(H_1, H_2)$ . Let  $K \in K(H_1, H_2)$  and  $K' \in K(H_2, H_1)$  be two compact operators and consider  $S_1 = S + K$  and  $T_1 = T + K'$ . Suppose that  $R(T_1 S_1)$  and  $R(S_1 T_1)$  are finite dimensional subspaces of  $H_1$  and  $H_2$  respectively. Then,  $(S_1, T_1)$  is a Fredholm pair. Furthermore,*

$$\text{ind}(S, T) = \text{ind}(S_1, T_1).$$

*Proof.* First of all, according to Theorem 4.2,  $S + T^* \in L(H_1, H_2)$  is a Fredholm operator and  $\text{ind}(S, T) = \text{ind}(S + T^*)$ .

Now well, since

$$S_1 + T_1^* = (S + K) + (T + K')^* = (S + T^*) + (K + K'^*),$$

$S_1 + T_1^*$  is a Fredholm operator and  $\text{ind}(S + T^*) = \text{ind}(S_1 + T_1^*)$ .

However, according to Theorem 4.2,  $(S_1, T_1)$  is a Fredholm pair and

$$\text{ind}(S, T) = \text{ind}(S_1, T_1). \quad \square$$

## 6. FREDHOLM CHAINS

In this section, as an application of the main results of this article, characterizations of Fredholm chains in Hilbert spaces are given. Furthermore, the stability properties and the relationship between adjoint operators and the objects under consideration are studied. For a general exposition, see [6].

In what follows, sequences of spaces and maps  $(H, \delta)$  and the operators  $S$  and  $T$  associated to such a sequence will be considered, see Remark 2.4. In addition, in order to lighten the text, the spaces  $H_p$  and the maps  $\delta_p$  will be indexed for  $p \in \mathbb{Z}$ . However, recall that there is always an  $n \in \mathbb{N}$  such that  $H_p = 0$  and  $\delta_p = 0$ , for  $p < 0$  and  $p \geq n+1$ .

In first place, two characterizations of Fredholm chains are given.

**THEOREM 6.1.** *Let  $(H, \delta)$  be a sequence of space and maps such that  $\dim R(\delta_p \delta_{p+1})$  is finite dimensional for each  $p \in \mathbb{Z}$ . Then, the following statements are equivalent:*

- i)  $(H, \delta)$  is a Fredholm chain,
- ii)  $\delta_{p+1} \delta_{p+1}^* + \delta_p^* \delta_p \in L(H_p)$  is a Fredholm self-adjoint operator for each  $p \in \mathbb{Z}$ .

*Proof.* As in Remark 2.4, consider the spaces

$$H_1 = \bigoplus_{p=2k} H_p, \quad H_2 = \bigoplus_{p=2k+1} H_p,$$

and the maps  $S \in L(H_1, H_2)$  and  $T \in L(H_2, H_1)$  defined by

$$S = \bigoplus_{p=2k} \delta_p, \quad T = \bigoplus_{p=2k+1} \delta_p.$$

According to Remark 2.4 and to Theorem 3.5,  $(H, \delta)$  is a Fredholm chain if and only if  $TT^* + S^*S \in L(H_1)$  and  $SS^* + T^*T \in L(H_2)$  are Fredholm self-adjoint operators, which is equivalent to the second statement of the theorem.  $\square$

**THEOREM 6.2.** *Let  $(H, \delta)$  be a sequence of space and maps such that  $\dim R(\delta_p \delta_{p+1})$  is finite dimensional for each  $p \in \mathbb{Z}$ . Then, the following statements are equivalent:*

- i)  $(H, \delta)$  is a Fredholm chain,
- ii)  $\bigoplus_{p=2k} (\delta_p + \delta_{p+1}^*)$  is a Fredholm operator,
- ii)  $\bigoplus_{p=2k+1} (\delta_p + \delta_{p+1}^*)$  is a Fredholm operator.

Furthermore, in this case

$$\text{ind } (H, \delta) = \text{ind } \bigoplus_{p=2k} (\delta_p + \delta_{p+1}^*) = - \text{ind } \bigoplus_{p=2k+1} (\delta_p + \delta_{p+1}^*).$$

*Proof.* As in Theorem 6.1, consider the spaces and maps  $H_1, H_2, S$  and  $T$  defined in Remark 2.4.

According to Remark 2.4 and to Theorem 4.2,  $(H, \delta)$  is a Fredholm chain if and only if  $S + T^*$  is a Fredholm operator, which is equivalent to the second statement of the theorem.

Similarly,  $(H, \delta)$  is a Fredholm pair if and only if  $T + S^*$  is a Fredholm pair, which is equivalent to the third statement of the theorem.

Finally, the index formula is a consequence of Remark 2.4 and Theorem 4.2.  $\square$

Next, the dual of a Fredholm chain is considered.

*Definition 6.3* Let  $(H, \delta)$  be sequence of spaces and maps

$$0 \rightarrow H_n \xrightarrow{\delta_n} H_{n-1} \rightarrow \dots \rightarrow H_1 \xrightarrow{\delta_1} H_0 \rightarrow 0,$$

where  $H_p$  are Hilbert spaces, and  $\delta_p \in L(H_p, H_{p-1})$  are bounded operators. In addition, assume that  $H_p = 0$  and  $\delta_p = 0$ , for  $p < 0$  and  $p \geq n+1$ , where  $n$  is the first natural number with this property. The dual sequence of  $(H, \delta)$  is the sequence  $(H', \delta')$ , where  $H'_p = H_{n-p}$  and  $\delta'_p = \delta_{n-p+1}^*$ , that is the sequence

$$0 \rightarrow H_0 \xrightarrow{\delta_1^*} H_1 \rightarrow \dots \rightarrow H_{n-1} \xrightarrow{\delta_n^*} H_n \rightarrow 0.$$

**THEOREM 6.4.** *Let  $(H, \delta)$  be a sequence of spaces and maps, and consider its dual sequence  $(H', \delta')$ . Then,  $(H, \delta)$  is a Fredholm chain if and only if  $(H', \delta')$  is. Furthermore, in this case*

$$\text{ind}(H, \delta) = (-1)^n \text{ind}(H', \delta'),$$

where  $n$  is the first natural number such that  $H_p = 0$  for  $p \geq n+1$ .

*Proof.* As in Remark 2.4, consider the spaces and maps defined by  $(H, \delta)$ , that is  $H_1, H_2, S$  and  $T$ . Similarly, consider the spaces and maps defined by  $(H', \delta')$ , that is  $H'_1, H'_2, S'$  and  $T'$ .

Now well, if  $n$  is an even natural number, then, according to Remark 2.4,  $H'_1 = H_1, H'_2 = H_2, S' = T^*$  and  $T' = S^*$ . Therefore, according to Remark 2.4 and to Theorem 4.3,  $(H, \delta)$  is a Fredholm chain if and only if  $(H', \delta')$  is, and in this case  $\text{ind}(H, \delta) = \text{ind}(H', \delta')$ .

On the other hand, if  $n$  is an odd natural number, then, according to Remark 2.4,  $H'_1 = H_2, H'_2 = H_1, S' = S^*$  and  $T' = T^*$ . Consequently, according to Remarks 2.2 and 2.4, and to Theorem 4.3,  $(H, \delta)$  is a Fredholm chain if and only if  $(H', \delta')$  is, and in this case  $\text{ind}(H, \delta) = -\text{ind}(H', \delta')$ .  $\square$

In the following theorems, the stability properties of Fredholm chains in Hilbert spaces are considered.

**THEOREM 6.5.** *Let  $(H, \delta)$  be a Fredholm chain and  $(H, \delta')$  a sequence of space and maps such that  $R(\delta'_p \delta'_{p+1})$  is finite dimensional for each  $p \in \mathbb{Z}$ . Then, there is an  $\epsilon > 0$  such that if  $\|\delta_p - \delta'_p\| < \epsilon, p \in \mathbb{Z}$ , then,  $(H, \delta')$  is a Fredholm pair. Furthermore,*

$$\text{ind}(H, \delta) = \text{ind}(H, \delta').$$

*Proof.* As in Theorems 6.1 and 6.2, consider the spaces and maps  $H_1, H_2, S$  and  $T$  associated to  $(H, \delta)$  and defined in Remark 2.4. Similarly, consider the maps  $S'$  and  $T'$  associated to  $(H, \delta')$ .

Since

$$\|S - S'\| \leq \sum_{p=2k} \|\delta_p - \delta'_p\|, \quad \|T - T'\| \leq \sum_{p=2k+1} \|\delta_p - \delta'_p\|,$$

according to Theorem 5.1, there is an  $\epsilon > 0$  such that if  $\|\delta_p - \delta'_p\| < \epsilon$ ,  $p \in \mathbb{Z}$ , then  $(S', T')$  is a Fredholm pair. Therefore, according to Remark 2.4, under this assumption  $(H, \delta')$  is a Fredholm chain.

Finally, the index formula is a consequence of Remark 2.4 and Theorem 5.1.  $\square$

**THEOREM 6.6.** *Let  $(H, \delta)$  be a Fredholm chain, and let  $k_p$  belong to  $K(H_p, H_{p-1})$ ,  $p \in \mathbb{Z}$ . Consider the sequences of spaces and maps  $(H, \delta')$ , where  $\delta'_p = \delta_p + k_p$ ,  $p \in \mathbb{Z}$ , and suppose that  $R(\delta'_p \delta'_{p+1})$  is finite dimensional for each  $p \in \mathbb{Z}$ . Then,  $(H, \delta')$  is a Fredholm chain. Furthermore,*

$$\text{ind}(H, \delta) = \text{ind}(H, \delta')$$

*Proof.* As in Theorem 6.5, consider the spaces and maps  $H_1, H_2, S, T, S'$  and  $T'$ .

It is clear that  $S' = S + K$  and  $T' = T + K'$ , where

$$K = \oplus_{p=2k} k_p, \quad K' = \oplus_{p=2k+1} k_p.$$

Therefore, according to Theorem 5.2 and to Remark 2.4,  $(H, \delta')$  is a Fredholm chain.

Finally, the index formula is a consequence of Remark 2.4 and Theorem 5.2.  $\square$

**Acknowledgments.** The author wishes to express his indebtedness to Professors C.-G. Ambrozie and F.-H. Vasilescu. These researchers sent kindly to the author several works authored by them, which have been useful for the elaboration of the present article.

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